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GALERKIN METHODS IN CIRCULAR AND SPHERICAL REGIONS

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by

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ABSTRACT

In this paper, it is shown how the convergence results for two-point (initial) boundary value problems, such as super convergence at the knots and invariance of convergence order when a proper quadrature rule is used, can be extended to more dimensions provided that circular symmetry exists.

KEY WORDS & PHRASES: *Finite element method, Galerkin method, spherical symmetry, two-point boundary value problems*

1. INTRODUCTION

In this paper, we want to solve numerically the N-dimensional boundary value problem

$$- \Delta u + q(x)u = - \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} + q(x)u = f(x), \; x \in I = [\emptyset,1];$$

(1.1) $$x = [\sum_{i=1}^{N} x_i^2]^{\frac{1}{2}};$$

$$u = \emptyset, \; x = 1.$$

Since u and f only depend on x, the problem easily reduces to the two-point boundary value problem

$$- x^{-N+1} \frac{d}{dx} (x^{N-1} \frac{du}{dx}) + q(x)u = f(x), \; \emptyset \le x \le 1;$$

(2.1) $$\frac{du}{dx} (\emptyset) = u(1) = \emptyset.$$

Note that the left boundary condition stems from the circular symmetry of u. In the sequel, we replace N-1 by C, where C is a non-negative integer.

In §2, we will show how problem (1.2) can be solved by the Galerkin method and how accurately this can be done. In §3 we develop (and advocate) some practical algorithms for C = 1 and C = 2. In §4, we show how parabolic equations can be semi-discretized to an explicit system of boundary differential equations. Finally, in §5 we give two simple numerical examples.

2. GALERKIN'S METHOD

Let $\varphi \in V = \{v | v \in C^{\emptyset}(I); \; v(1) = \emptyset\}$. If we multiply both sides of equation (1.2) by $x^C \varphi(x)$, we obtain after partial integration

(2.1) $$\int_0^1 x^C (\frac{du}{dx} \frac{d\varphi}{dx} + q(x) u \varphi) \, dx = \int_0^1 x^C f(x) \varphi dx.$$

In a way, formula (2.1) is a generalization of the weak Galerkin form (see STRANG & FIX [6]) for Cartesian two-point boundary value problems ($C = \emptyset$). We now define a suitable finite element space in which u can be approximated.
Let

$$(2.2) \qquad \Delta : \emptyset = x_1 < x_1 < \ldots < x_N = 1$$

be a partition of I, not necessarily uniform; let

$$I_j = [x_{j-1}, x_j];$$

$$(2.3) \qquad \Delta_j = x_j - x_{j-1}; \quad j = 1,\ldots,N;$$

$$|\Delta| = \max \Delta_j .$$

We assume that the knots x_j are chosen such that they coincide with any possible points where f(x) or q(x) are less smooth. Let k be some constant natural number and define for any segment $E \subset I$ $P_k(E)$ as the set of polynomials of degree less than or equal k restricted to E. Next, we define $M_\emptyset^k(\Delta)$ by

$$(2.4) \qquad M_\emptyset^k(\Delta) = \{\varphi | \varphi \in C^\emptyset(I);\ \varphi(1) = \emptyset;\ \varphi \in P_k(I_j), \quad j = 1,\ldots,N\}.$$

It is easily verified that $M_\emptyset^k(\Delta)$ is a kN-dimensional subspace of V. Now the finite element solution $U \in M_\emptyset^k(\Delta)$ of (1.2) is the solution of

$$(2.5) \qquad \left(\frac{dU}{dx}, \frac{d\varphi}{dx}\right) + (qU,\varphi) = (f,\varphi), \ \varphi \in M_\emptyset^k(\Delta),$$

where the inner product (α,β) is defined by

$$(2.6) \qquad (\alpha,\beta) = \int_0^1 x^C \alpha(x)\ \beta(x)dx, \quad \alpha_1\ \beta \in L^2(I).$$

After this definition of the Galerkin or finite element solution of (1.2) for spherical coordinates, we could proceed by formulating and proving convergence theorems. However, that would be merely consist of copying existing theorems, since the only difference would lie in the definition of the $L^2(I)$ inner product and of the appearing $H^m(I)$ norms and partition norms. Hence, instead of proving them anew, we refer to the papers where the proofs can be found for $C = \emptyset$.

THEOREM 1. *Let* $f(x)$ *and* $q(x)$ *be such that the solution* u *of* (1.2) *is in* $C^k_\emptyset(\Delta) = V \cap C^k(I_1) \cap \ldots \cap C^k(I_N)$; *let* $U \in M^k_\emptyset(\Delta)$ *be the solution of* (2.5) *and let* $E(x) = u(x) = U(x)$ *be the error function. Then*

$$\|E\|_\ell = O(|\Delta|^{k+1-\ell}\ \|u\|_{\Delta,k+1}),\quad \ell = \emptyset,1; \tag{2.7}$$

$$|E(x_i)| = O(|\Delta|^{2k}\ \|u\|_{\Delta,k+1}),\quad i = \emptyset,\ldots,N-1; \tag{2.8}$$

where $\|\cdot\|_\ell$ *and* $\|\cdot\|_{\Delta,m}$ *are defined by*

$$\|\alpha\|_\ell = [\sum_{j=\emptyset}^{\ell} (\frac{d^j\alpha}{dx^j}, \frac{d^j\alpha}{dx^j})]^{\frac{1}{2}};$$

$$\|\alpha\|_{\Delta,m} = [\sum_{\ell=1}^{N}\sum_{j=\emptyset}^{m} (\frac{d^j\alpha}{dx^j}, \frac{d^j\alpha}{dx^j})_{L^2(I_\ell)}]^{\frac{1}{2}}; \tag{2.9}$$

$$(\alpha,\beta)_{L^2(I_\ell)} = \int_{x_{\ell-1}}^{x_\ell} x^C\ \alpha(x)\ \beta(x)dx,\quad \alpha,\ \beta \in L^2(I_\ell),\quad \ell = 1,\ldots,N.$$

PROOF. See STRANG & FIX [6] for (2.7) and DOUGLAS & DUPONT [3] for (2.8). □

3. NUMERICAL QUADRATURE

To solve (2.5) numerically, the inner products (qU,φ) and (f,φ) have to be computed by some quadrature rule. As DOUGLAS et al. [3] and HEMKER [4] pointed out, the choice of that quadrature rule is strongly determined by the kind of finite element space in which $u(x)$ is approximated. In this § we will device some algorithms for $k = 1$ and $k = 2$.

3.1. Preservation of accuracy

We recall that

$$(\alpha,\beta)_{I_j} = \int_{x_{j-1}}^{x_j} x^C \alpha(x)\beta(x)dx, \quad \alpha,\beta \in L^2(I_j), \quad j = 1,\dots,N.$$

$$(\alpha,\beta) = \sum_{j=1}^{N} (\alpha,\beta)_{I_j}, \quad \alpha,\beta \in L^2(I).$$

Now let

$$\text{(3.1)} \qquad \langle\alpha,\beta\rangle_j = \Delta_j \sum_{\ell=r}^{s} w_{j,\ell}\, \alpha(\xi_{j,\ell})\beta(\xi_{j,\ell})$$

be some approximation of $(\alpha,\beta)_{I_j}$ which is exact if $\alpha\,\beta \in P_{2k-1}(I_j)$, with positive $w_{j,\ell}$ and $\xi_{j,\ell} \in I_j$, and define $\langle\alpha,\beta\rangle$ by

$$\text{(3.2)} \qquad \langle\alpha,\beta\rangle = \sum_{j=1}^{N} \langle\alpha,\beta\rangle_j .$$

For $C = \emptyset$, examples of such quadrature are k-point Gauss-Legendre and (k+1)-point Lobatto shifted to the interval I_j.

THEOREM 2. *Let* $f(x)$ *and* $q(x)$ *be such that the solution* u *of* (1.2) *is in* $C_\emptyset^{2k-1}(\Delta)$ *and let* $U \in M_\emptyset^k(\Delta)$ *be defined by*

$$\text{(3.3)} \qquad \langle U',\varphi'\rangle + \langle qU,\varphi\rangle = \langle f,\varphi\rangle, \quad \varphi \in M_\emptyset^k(\Delta).$$

Then $E(x) = u(x) - U(x)$ *has the following bounds*

$$\text{(3.3)} \qquad \begin{aligned} \|E\|_\ell &= O(|\Delta|^{k+1-\ell}\|u\|_{\Delta,2k}), \quad \ell = \emptyset,1; \\ |E(x_j)| &= O(|\Delta|^{2k}\|u\|_{\Delta,2k}), \quad j = \emptyset,\dots,N-1. \end{aligned}$$

PROOF. See DOUGLAS et al. [3]. □

3.2. Construction of some algorithms

As was remarked in the previous section, for C = ∅ k-point Gauss-Legendre and (k+1)-point Lobatto quadrature (see HEMKER [4]) are suitable quadrature rules to solve (3.3) numerically. We want to generalize both rules for $C > 0$. We note however that, contrary to the case $C = 0$, practical algorithms for finite element spaces of degree greater than 2 are hardly feasible.

3.2.1. Gaussian quadrature

The basic problem is to find an approximation for $(\alpha,\beta)_{I_j}$ of the form

$$(3.4)\qquad \langle\alpha,\beta\rangle_j = \Delta_j \sum_{\ell=1}^{k} w_{j,\ell}\, \alpha(\xi_{j,\ell})\beta(\xi_{j,\ell}),$$

where $w_{j,\ell}$ are positive weights and $\xi_{j,\ell}$ are *interior* points of I_j.

This problem can be solved by applying the theory of Gaussian quadrature (see DAVIS & RABINOWITZ [2]). Let $\{\psi_{j,i}\}_{i=0}^{k}$ be a set of polynomials orthonormal on I_j with respect to the weight function x^C, i.e.

$$(3.5)\qquad (\psi_{j,i},\psi_{j,\ell})_{L^2(I_j)} = \delta_{i,\ell}; \quad \emptyset \le i,\ \ell \le k,$$

where $\delta_{i\ell}$ is the Kronecker symbol. Then $\xi_{j,\ell}$ is the ℓ-th zero of $\psi_{j,k}(x)$ and $w_{j,\ell}$ is given by

$$(3.6)\qquad w_{j,\ell} = [\sum_{i=0}^{k-1} \psi_i^2(\xi_{j,\ell})]^{-1}, \quad \ell = 1,\dots,k.$$

For $k = 1$ and C arbitrary, the solution is

$$\xi_{j,1} = \frac{C+1}{C+2}\,\frac{x_j^{C+2}-x_{j-1}^{C+2}}{x_j^{C+1}-x_{j-1}^{C+1}}$$

$$= \frac{C+1}{C+2}\,\frac{x_{j-1}^{C+1}+x_{j-1}^{C}x_j+\ldots+x_j^{C+1}}{x_{j-1}^{C}+x_{j-1}^{C-1}x_j+\ldots+x_j^{C}}\ ;$$

$$w_{j,1} = \frac{x_j^{C+1}-x_{j-1}^{C+1}}{(C+1)(x_j-x_{j-1})} = \frac{x_{j-1}^{C}+x_{j-1}^{C-1}x_j+\ldots+x_j^{C}}{C+1}\ ,$$

For $k > 1$ the weights and abscissae are more difficult to compute, hence we only give two examples for $k = 2$.

<u>C = 1; k = 2</u>

(3.9)

$$\xi_{j,\ell} = \frac{6P_3(x_{j-1},x_j)\pm\Delta_j\sqrt{6P_4(x_{j-1},x_j)}}{10P_2(x_{j-1},x_j)}\ , \quad \ell = 1,2;$$

$$w_{j,\ell} = \tfrac{1}{4}(x_{j-1}+x_j) \pm \frac{\Delta_j R_2(x_{j-1},x_j)}{6\sqrt{6P_4(x_{j-1},x_j)}}\ , \quad \ell = 1,2;$$

where

$$P_2(a,b) = a^2 + 4ab + b^2;$$

$$R_2(a,b) = a^2 + 7ab + b^2;$$

$$P_4(a,b) = a^4 + 10(a^3b + ab^3) + 28a^2b^2 + b^4;$$

$$P_3(a,b) = a^3 + 4(a^2b + ab^2) + b^3.$$

<u>C = 2; k = 2</u>

(3.10)

$$\xi_{j,\ell} = \frac{10P_5(x_{j-1},x_j)\pm\Delta_j\sqrt{10P_8(x_{j-1},x_j)}}{15P_4(x_{j-1},x_j)}$$

$$w_{j,\ell} = \frac{1}{6}\,P_2(x_{j-1},x_j) \pm \frac{5}{24}\,\Delta_j\,\frac{R_5(x_{j-1},x_j)}{\sqrt{10P_8(x_{j-1},x_j)}}\ , \quad \ell = 1,2;$$

where

$$P_2(a,b) = a^2 + ab + b^2;$$

$$P_5(a,b) = a^5 + 4(a^4b + ab^4) + 10(a^3b^3 + a^2b^3) + b^5;$$

$$R_5(a,b) = a^5 + 7(a^4b + ab^4) + 28(a^3b^2 + a^2b^3) + b^5;$$

$$P_8(a,b) = a^8 + 10(a^7b+ab^7) + 55(a^6b^2 + a^2b^6)$$

$$+ 164(a^5b^3 + a^3b^5) + 290a^4b^4 + b^8.$$

3.2.2. Lobatto quadrature

The problem is to approximate $(\alpha,\beta)_{I_j}$ by a formula of the form

(3.11)
$$\langle\alpha,\beta\rangle = \Delta_j \sum_{\ell=0}^{k} w_{j,\ell}\, \alpha(\xi_{j,\ell})\beta(\xi_{j,\ell});$$
$$x_{j-1} = \xi_{j,0} < \xi_{j,1} < \dots < \xi_{j,k} + x_j;\ w_{j,\ell} > 0$$

which is exact if $\alpha\,\beta \in P_{2k-1}(I_j)$. This problem can be solved by the theory of Gaussian quadrature either. Let $\{\psi_{j,i}\}_{i=0}^{k-1}$ be a set of polynomials orhonormal on I_j with respect to the weight function $(x - x_{j-1})(x_j - x)x^C$, i.e.

(3.12)
$$\int_{x_{j-1}}^{x_j} (x-x_{j-1})(x_j-x)x^C\ \psi_{j,i}(x)\ \psi_{j,\ell}(x)dx = \delta_{i,\ell},\ 0 \le i,\ \ell \le k-1.$$

Then, if $k > 1$, the abscissae $\xi_{j,1},\dots,\xi_{j,k-1}$ are the zeros of $\psi_{k-1,j}(x)$. The weights are now easily found by applying (3.11) to $\alpha(x) \equiv 1$ and $\beta(x) = \varphi_{j,\ell}(x)$, where $\varphi_{j,\ell}(x)$ is a polynomial of degree k (a so-called Lagrance interpolation polynomial) defined by

$$\varphi_{j,\ell}(\xi_{j,m}) = \delta_{\ell,m};\ 0 \le \ell,\ m \le k.$$

It then appears that

(3.13)
$$\Delta_j\, w_{j,\ell} = \int_{x_{j-1}}^{x_j} x^C\ \varphi_{j,\ell}(x)dx,\ \ell = 0,\dots,k.$$

For k = 1 and C arbitrary, the solution of (3.11) is

$$w_{j,0} = \frac{(C+1)x_{j-1}^{C} + Cx_{j-1}^{C-1}x_j + \ldots + x_j^{C}}{(C+1)(C+2)} ;$$

(3.14)

$$w_{j,1} = \frac{x_{j-1}^{C} + 2x_{j-1}^{C-1}x_j + \ldots + (C+1)x_j^{C}}{(C+1)(C+2)} .$$

For k > 1, as in the Gauss-Legendre case, the weights and abscissae are more difficult to find. Hence we list only two examples.

C = 1; k = 2

$$\xi_{j,1} = x_{j-1} + \Delta_j\left(\tfrac{1}{2} + \frac{\Delta_j}{10(x_{j-1}+x_j)}\right)$$

$$w_{j,0} = \frac{3x_{j-1}^2 + 6x_{j-1}x_j + x_j^2}{12(2x_{j-1}+3x_j)} ;$$

(3.15)

$$w_{j,1} = \frac{25(x_{j-1}+x_j)^3}{12(2x_{j-1}+3x_j)(3x_{j-1}+2x_j)}$$

$$w_{j,2} = \frac{x_{j-1}^2 + 6x_{j-1}x_j + 3x_j^2}{12(3x_{j-1}+2x_j)} ;$$

C = 2; k = 2

$$\xi_{j,1} = x_{j-1} + \Delta_j\left[\tfrac{1}{2} + \frac{\Delta_j(x_{j-1}+x_j)}{2P_2(x_{j-1},x_j)}\right];$$

$$w_{j,0} = \frac{P_4(x_{j-1},x_j)}{60R_2(x_{j-1},x_j)} ;$$

(3.16)

$$w_{j,1} = \frac{[P_2(x_{j-1},x_j)]^3}{60R_2(x_{j-1},x_j)R_2(x_j,x_{j-1})} ;$$

$$w_{j,2} = \frac{P_4(x_j,x_{j-1})}{60R_2(x_j,x_{j-1})} ;$$

where

$$P_2(a,b) = 3a^2 + 4ab + 3b^2;$$

$$R_2(a,b) = 2a^2 + 2ab + b^2;$$

$$P_4(a,b) = 6a^4 + 16a^3b + 21a^2b^2 + 6ab^3 + b^4.$$

One final remark about the use of Lobatto quadrature. As HEMKER [4] has already proved for C = 0, the basis functions of $M_\emptyset^k(\Delta)$ can be selected such that they form a system orthogonal with respect to the Lobatto quadrature; by this we mean the following: let the set of points $\{z_n\}_{n=0}^{kN-1}$ be given by

(3.18)
$$z_{\ell k} = x_\ell, \quad \ell = 0,\dots,N$$
$$z_{\ell k+i} = \xi_{\ell+1,i}, \quad \ell = 0,\dots,N-1; \quad i = 1,\dots,k-1,$$

where $\xi_{j,i}$ are determined by (3.13)-(3.17). We define $\varphi_\emptyset(x),\dots,\varphi_{kN-1}(x)$ by

(3.19) $$\varphi_i(z_\ell) = \delta_{i,\ell}, \quad \emptyset \le i, \ \ell \le kN-1.$$

Now it is easily checked that

(3.20)
$$<qU,\varphi > = \lambda_i \ q(z_i) \ U(z_i);$$
$$<f \ ,\varphi_i> = \lambda_i \ f(z_i);$$
$$<\varphi_i \ \varphi_j> = \lambda_i \ \delta_{ij};$$
$$\lambda_i = <\varphi_i,\varphi_i>; \quad \emptyset \le i, \ j \le kN-1.$$

Hence if we write

$$U(x) = \sum_{i=0}^{kN-1} a_i \ \varphi_i(x),$$

it turns out that $(a_\emptyset,\dots,a_{kN-1})^T$ is the solution of

$$(3.21) \qquad \sum_{j=0}^{kN-1} \langle \varphi_i', \varphi_j' \rangle a_j + \lambda_i \overline{q}(z_i) a_i = \lambda_i \overline{f}(z_i),$$

where for any $\alpha \in L^2(I)$ $\overline{\alpha}(x)$ is defined by

$$(3.22) \qquad \overline{\alpha}(x) = \begin{cases} \alpha(x), & x \neq x_j, \quad j = 1,\dots,N-1; \\ \dfrac{w_{j,k}\alpha_-(x_j) + w_{j+1,\emptyset}\alpha_+(x_j)}{w_{j,k} + w_{j+1,\emptyset}}, & x = x_j, \quad j = 1,\dots,N-1, \end{cases}$$

$$\alpha_-(x_j) = \lim_{x \uparrow x_j} \alpha(x); \quad \alpha_+(x_j) = \lim_{x \downarrow x_j} \alpha(x), \quad j = 1,\dots,N-1,$$

which is an easily implementable algorithm, once the Lobatto weights have been computed. Note that the matrix $(\langle \varphi_i', \varphi_j' \rangle)$ is $(2k+1)$-diagonal.

4. INITIAL BOUNDARY VALUE PROBLEMS

We consider the differential equation

$$(4.1) \qquad \frac{\partial u}{\partial t} = x^{-C} \frac{\partial}{\partial x} \left(x^C \frac{\partial u}{\partial x}\right) - q(x)u + f(x), \quad \emptyset < x < 1,$$

with boundary conditions

$$(4.2) \qquad \frac{\partial u}{\partial x}(\emptyset, t) = u(1,t) = \emptyset$$

and initial conditions

$$(4.3) \qquad u(x,\emptyset) = v(x).$$

We assume that $v \in V$ and that q, f and v are sufficiently smooth.

Again, let $\Delta: 0 = x_0 < x_1 < \dots < x_N = 1$ be a grid of I such that all interior points of I where q, f and v are less smooth are contained in Δ. Let $M_\emptyset^k(\Delta)$ be defined by (2.4) for some constant k. Then one easily sees that the relation

$$(4.4) \qquad \left(\frac{\partial u}{\partial t}, \varphi\right) + \left(\frac{\partial u}{\partial x}, \frac{\partial \varphi}{\partial x}\right) + (qu, \varphi) = (f, \varphi), \quad \varphi \in M_\emptyset^k(\Delta)$$

holds. As is well-known, an approximation for $u(x,t)$ in $M^k_\emptyset(\Delta)$ can be found by restricting (4.4) to $M^k_\emptyset(\Delta)$ and by approximations $v(x)$ in $M^k_\emptyset(\Delta)$ properly.

THEOREM 3. *Let* $U\colon [\emptyset,\infty] \to M^k_\emptyset(\Delta)$ *be the solution of the initial value problem (in Galerkin form)*

$$(4.5)\qquad \begin{aligned} &(\frac{\partial}{\partial t} U,\varphi) + (\frac{\partial U}{\partial x}, \frac{\partial \varphi}{\partial x}) + (qU,\varphi) = (f,\varphi),\ \varphi \in M^k_\emptyset(\Delta);\ t \geq \emptyset \\ &(U(\,,\emptyset),\varphi) = (v,\varphi),\ \varphi \in M^k_\emptyset(\Delta); \end{aligned}$$

and let $u \in C^k_0(\Delta)$ *be the solution of* (4.1)-(4.3). *Then, the error function* $E(x,t) = u(x,t) - U(x,t)$ *has the following bounds*

$$(4.6)\qquad \begin{aligned} &\|E(\,,t)\|_\emptyset = O(|\Delta|^{k+1}); \\ &|E(x_j,t)| = O(|\Delta|^{2k}); \qquad j = 0,\dots,N-1. \end{aligned}$$

PROOF. See BAKKER [1]. □

Now it is obvious that one should try to apply one of the quadrature rules from §3 to (4.6). There is, however, one problem. One needs the property that the "inner product" $<\alpha,\beta>$ induces a norm on $M^k_0(\Delta)$ equivalent to the $L^2(I)$ norm. The Lobatto rule < > has this property (see BAKKER [1]), the Gauss-Legendre rule < > not. So, at first, we introduce different notations for the generalized Lobatto and Gauss-Legendre quadrature rules. The former is denoted by $<\,,\,>_L$, the latter by $<\,,\,>_G$.

THEOREM 4. *Let* $U : [\emptyset,\infty] \to M^k_\emptyset(\Delta)$ *be the solution of the differential equation*

$$(4.7)\qquad \begin{aligned} &<\frac{\partial}{\partial t}U,\varphi>_L + <\frac{\partial}{\partial x}U, \frac{\partial}{\partial x}\varphi>_L + <qU,\varphi>_L = <f,\varphi>_L; \\ &<U(\,,\emptyset),\varphi>_L = <v,\varphi>_L;\ \varphi \in M^k_\emptyset(\Delta). \end{aligned}$$

and let $u : [0,\infty) \to C_0^{2k-1}(\Delta)$ *be the solution of* (4.1). *Then the error function* $E(x,t) = u(x,t) - U(x,t)$ *has the bounds.*

$$\|E(,t)\|_0 = O(|\Delta|^{k+1});$$

(4.8)

$$|E(x_j,t)| = O(|\Delta|^{2k}); \qquad j = 0,\ldots,N-1.$$

PROOF. See BAKKER [1]. □

After this theorem, we evaluate the resulting O.D.E. If we represent U(x,t) by

$$(4.9) \qquad U(x,t) = \sum_{j=0}^{kN-1} a_j(t)\,\varphi_j(x),$$

where the basis $\{\varphi_j(x)\}_{j=1}^{kN-1}$ is defined by (3.18)-(3.19), we easily verify, by combining (3.18)-(3,20) with (4.7) that the vector $(a_0,a_1,\ldots,a_{kN-1})$ satisfies the O.D.E.

$$\frac{d}{dt} a_i = -\lambda_i^{-1} \sum_{j=0}^{kN-1} \langle\varphi_i',\varphi_j'\rangle a_j - \overline{q}(z_i)a_i + \overline{f}(z_i)$$

(4.10)

$$a_i(0) = v(z_i), \quad i = 0,\ldots,kN-1.$$

5. NUMERICAL EXAMPLES

In order to demonstrate the use of Galerkin methods in spherical regions, we solved two simple problems: a boundary value problem and an initial boundary value problem.

5.1. Problem A

We consider the boundary value problem: find the solution $u \in C'(I)$ of

$$(5.1) \qquad -\frac{1}{x}\frac{d}{dx}\left(x\frac{du}{dx}\right) = \begin{cases} 1, & \emptyset \le x < \frac{1}{2}; \\ 2, & \frac{1}{2} < x \le 1; \end{cases}$$

$$u'(\emptyset) = u(1) = \emptyset.$$

The exact solution is given by

$$u(x) = \begin{cases} \frac{7}{16} - \frac{1}{8}\ell n(z) - \frac{x^2}{4} & , \emptyset < x \le \frac{1}{2}; \\ \frac{1}{2}(1-x^2) + \frac{1}{8}\ell n(x) & , \frac{1}{2} < x < 1. \end{cases}$$

Note that $u''(x)$ does not exist for $x = \frac{1}{2}$ but that $u \in C'(I) \cap C^{\infty}(\emptyset,\frac{1}{2}) \cap C^{\infty}(\frac{1}{2},1)$. We divided I into 1∅ and 2∅ segments of equal length. In both cases $x = \frac{1}{2}$ was one of the mesh-points. After that, we solved (5.1) by means of Galerkin's method for k = 1 and k = 2. In table I, we list the maximum error at the gridpoints.

N \ K	1	2
1∅	$6.28_{10}{-3}$	$2.54_{10}{-7}$
2∅	$2.22_{10}{-3}$	$1.62_{10}{-8}$

Table I; $\max_{i=\emptyset,\dots,N-1} |E(x_i)|$.

5.2. Problem B

We consider the initial boundary value problem

$$\frac{\partial u}{\partial t} = \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) - x^2 u + x(1-x),\ x \in I,\ t \ge \emptyset$$

$$(5.2) \qquad u_x(\emptyset,t) = u(1,t) = \emptyset$$

$$u(x,\emptyset) = \emptyset.$$

Problem (5.2) was semi-discretized uniformly to the ordinary differential equation (4.11) for k = 2, c = 1 and N = 10. For the semi-discretization we used generalized 3-point Lobatto quadrature. The resulting explicit O.D.E. was integrated by a fourth order Runge-Kutta method. With time steps of $_{10}-3$.

T\x	∅.∅	∅.2	∅.4	∅.6	∅.8
∅.5	$4.4219_{10}-2$	$4.3606_{10}-2$	$3.9399_{10}-2$	$2.9923_{10}-2$	$1.5835_{10}-2$
1.0	$4.6787_{10}-2$	$4.6023_{10}-2$	$4.1391_{10}-2$	$3.1296_{10}-2$	$1.6507_{10}-2$
1.5	$4.6922_{10}-2$	$4.6149_{10}-2$	$4.1495_{10}-2$	$3.1368_{10}-2$	$1.6542_{10}-2$
2.∅	$4.6922_{10}-2$	$4.6149_{10}-2$	$4.1495_{10}-2$	$3.1368_{10}-2$	$1.6542_{10}-2$

Table II; results of (5.2)

As a check, we solved the steady-state problem by means of the power series expansion

$$u_\infty(x) = \sum_{n=0}^{\infty} a_n x^n;$$

$$a_0 = A = 0.0469345729;$$

$$a_1 = a_2 = 0;$$

$$a_3 = -\frac{1}{9};$$

$$a_4 = \frac{1+A}{16}$$

$$a_n = \frac{a_{n-4}}{n^2}, \quad n \geq 5$$

and compared $u_\infty(x)$ with the steady-state values from table II. We found a maximum error of $1.2_{10}-5$.

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